INTRODUCTION
ANALYSIS OF EXPERIMENTAL DATA

CONTENTS

1. SIGNIFICANT FIGURES, ACCURACY AND PRECISION

Physical quantities are measured in the laboratory with the help of various instruments. These instruments have scales with divisions. The digits that are read and estimated on the scale are called *significant figures*.

Suppose the length AB of an object is measured with a meter stick (Fig. 1). The smallest subdivision on the meter stick is 1 cm. Point A is aligned with the 0 graduation mark and point B falls between the 7 and 8 graduation marks. Therefore the length AB is somewhere between 7 and 8 cm. A reasonable estimate is that B is located at 7.6 cm. Of

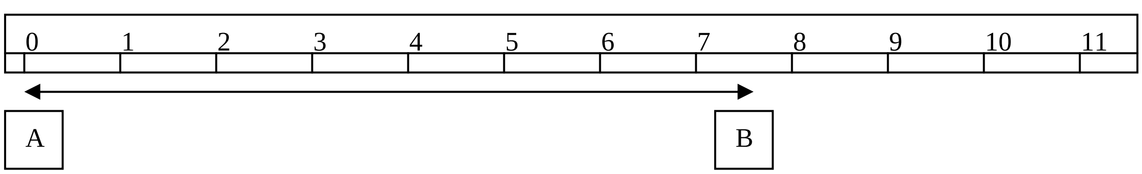

Figure 1.

those two digits the first one is certain, the last one is doubtful. The length AB can therefore be given to two significant figures. The digit preceding the doubtful digit represents in general the smallest subdivision on the scale. If the length AB had been recorded as 7.65 cm, the impression would have been given that the scale of the ruler was divided into a tenth of a centimeter and that the point B was located between the 7.6 and the 7.7 graduation mark. Thus the length AB would be closer to the value of 7.65 cm than to 7.64 or 7.66 cm. More information is therefore given and there are now three significant figures.

Very often in the process of calculations extra figures are accumulated and sometimes reported in the end result. This is a meaningless procedure**. No figures should be included beyond the precision of the original data.** In the case of random errors the root-mean-square error in the arithmetic mean (see following sections) determines the number of significant figures. As an example L = (2.56 ± 0.03) m and *not* L = (2.556 ± 0.034) m or L = (2.56 ± 0.034) m.

If a resistance of 105 Ω is written as 0.000105 MΩ, there are still only three significant figures. The resistance can namely also be written as 105 * 10^{-6} MΩ. A figure such as 25600 has five significant figures. It means that the true value is located somewhere between 25601 and 25599. There are only three significant figures if the figure had been reported as 256 * 10^2. In this case the number has been specified to the nearest hundred. Some other examples are

30.31	Four significant figures
35	Two significant figures
15.5	Three significant figures
20.000	Five significant figures
0.0001059	Four significant figures
5.0001050	Eight significant figures

In publications and reports the results of the measurements in the laboratory are given. However no measurement is ever made with absolute accuracy. For example, a number like the velocity of light, c, is an exact number. Rulers and stopwatches are used to measure the velocity of light. Since these instruments are not ideal and their scales cannot be read exactly, the resultant measured value of the velocity of light is not exact. The better the measuring instruments are, the more exact the resulting values of the velocity of light can be.

A distinction should be made between *accurate* and *precise* measurements. A precise measurement of the velocity of light can yield a number with more significant figures, such as c = 3.09035 * 10^8 m/s. However this result is obviously wrong, that is, the result is not accurate.

Precision is measured by the uncertainty in the end result and, for example, can be reported as follows

$$c = (2.997925 \pm 0.000003) * 10^8 \text{ m/s}$$

This means that if the measurement is repeated, the resulting value of c will have a certain probability to be located between 2.997928 * 10^8 m/s and 2.997922 * 10^8 m/s. The smaller the uncertainty the more precise the measurement is.

Accuracy gives the closeness of the experimental results to the actual or correct value of the physical quantity. In contrast the precision is the closeness with which the measurements agree with one another.

To be accurate the results should be precise, however a result that is precise does not need to be accurate.

2. ERRORS OF OBSERVATION

All measurements contain errors. A study of errors is therefore important as a step in finding ways to reduce them and also as a way of estimating the reliability of the final result of the experiment.

There are two kinds of errors that are related to observations or measurements:

1. *Systematic* errors
2. *Random* errors

Systematic errors are due to the wrong calibration or wrong construction of the instruments used in making the measurements. Also external conditions (such as temperature, humidity, magnetic fields) and observational errors can contribute to systematic errors. The results of the measurements are then consistently too large or too small. Careful planning of the experiment can in general eliminate systematic errors.

On the contrary random errors are due to no known cause, they are of a statistical nature. They can be thought of being due to a large number of independent causes, producing small fluctuations in the measurements, and can result in equally too large or too small values of the measured quantity.

If N measurements of the quantity x are made then the best estimate for x is the *average* or the mean of x_1, x_2, x_3………, x_N. That is

$$\bar{x} = \frac{\sum x_i}{N} \qquad (1)$$

The differences

$$d_i = x_i - \bar{x} \qquad (2)$$

are called the deviations and indicate how much the i^{th} measurement x_i differs from the average $\bar{x}$. The mean deviation is of course equal to zero. The standard deviation or root-mean-square deviation of the N measurements of x_i which characterizes the reliability of the measurements, is defined as

$$\sigma = \sqrt{\frac{\sum d_i^2}{N-1}} \qquad (3)$$

Finally the standard deviation of the mean or the *root-mean-square error* in the arithmetic mean of N measurements is given by

$$\alpha = \sqrt{\frac{\sum d_i^2}{N(N-1)}} \qquad (4)$$

Equation (4) characterizes the uncertainty in the mean $\bar{x}$ as being the best estimate for x.

Table 1 gives an application of the above definitions. As an example the length L of a table is measured ten times with a meter stick. The smallest sub-division on the scale of the meter stick is one tenth of a meter. The measurements are tabulated in the second column of the table. The arithmetic mean of the length L and therefore the best estimate of the length L of the table is then given by

$$\bar{L} = \frac{\sum L_i}{N} = 2.556 \text{ m}$$

The deviations are given in the third column and the standard deviation is obtained from the fourth column. Finally the end result is quoted with its mean-square-error.

Table 1

Measurement #	L_i (m)	Deviation d_i	d_i^2 *1000
1	2.52	-0.036	1.296
2	2.69	+0.134	17.956
3	2.46	-0.096	9.216
4	2.58	0.024	0.576
5	2.39	-0.166	27.556
6	2.41	-0.146	21.316
7	2.62	+0.064	4.096
8	2.66	+0.104	10.816
9	2.67	+0.114	12.996
10	2.56	+0.004	0.016
	$\bar{L} = \mathbf{2.556}$		$\sum d_i^2 = \mathbf{0.106}$
$\sigma = 0.1$ $\alpha = 0.03$ $\bar{L} = (\mathbf{2.56} \pm \mathbf{0.03})$ m			

Occasionally a single measurement from a set of measurements differs widely from the others, so that the experimenter is tempted to discard the measurement.
The average deviation

$$\bar{d} = \frac{\sum \left| x_i - \bar{x} \right|}{N} \qquad (5)$$

can be used in order to decide if an observation should be rejected or not. A simple rule to follow in such cases is

(a) Calculate the deviation d for all the data including the suspected one.
(b) If any deviation exceeds four times the average deviation as given by equation 5, the particular measurement can be rejected. The reason is that the probability of an error four times as large as the average error is about one in one thousand.

3. THEORY OF ERRORS

If a quantity x is measured a very large (infinite) number of times, the resulting values x_i are distributed at random, assuming that the experiment has been set up in such a way that systematic errors have been eliminated. A plot of the measured values, x_i, against the number of observations (frequency) of a particular value x_i yields a so-called frequency distribution curve for the observations.

There are a number of well-known frequency distributions. In the case of random errors in scientific measurements, the frequency curve is called the normal or *Gaussian* distribution (fig. 2).

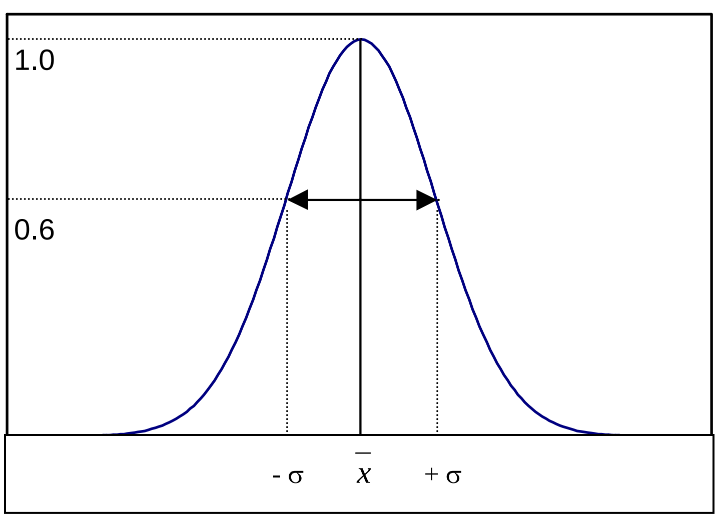

Figure 2.

This distribution is a bell-shape curve and can be presented by the normalized function

$$y = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(x-\bar{x})^2 / 2\sigma^2] \qquad (6)$$

$\bar{x}$ is the mean of the distribution and σ, which is a measure of the width of the curve, is the standard deviation of the distribution. The curve (fig. 2) is symmetric about the arithmetic mean or average value $\bar{x}$ of the measured values x_i and it can be shown that for x = $\bar{x} \pm \sigma$, y ≅ 0.607 times the maximum amplitude $(1/(\sigma\sqrt{2\pi})$ of the curve at x = $\bar{x}$. The area under the curve represents the probability that an observation will be falling into a given interval. The total area under the normalized curve is equal to one and 68% of the measured values, fall within one standard deviation σ of the mean, 95% fall within two standard deviations of the mean. Furthermore in a normal distribution half of the observations (50%) are located between $\bar{x} \pm 0.6745\ \sigma$.

4. PROPAGATION OF ERRORS

In order to estimate the error in *compound quantities*, the following procedure is followed. If a number of measured quantities have arithmetic means x, y, and z with root-mean-square errors of α_x, α_y, and α_z respectively, then the root-mean-square error α_F in any function F of x, y, and z is given by

$$\alpha_F = \sqrt{(\frac{\partial F}{\partial x}\alpha_x)^2 + (\frac{\partial F}{\partial y}\alpha_y)^2 + (\frac{\partial F}{\partial z}\alpha_z)^2} \qquad (8)$$

Example: Suppose that you want to calculate the density of a solid. You have measured its mass 5.0 ± 0.2 kg and its volume 2.00 ± 0.08 m^3. Calculate the density of the solid along with the error on its value.

5. LEAST-SQUARES FITTING OF A CURVE

Many experiments yield a series of pairs of data values. Usually the x_i values are selected and the y_i values are measured. A graph is plotted with each pair (x_i,y_i) representing a point. If the points of the graph are located within a narrow band, the variables x and y are said to be correlated, and a relation exists between x and y.

The *method of least squares* is used to fit a curve (find a theoretical equation) to a set of experimental data. First assume that a linear relation exists between y and x

$$y = Ax + B \tag{9}$$

Substitution of $x = x_i$ will in general not give the value of y_i. The "errors" will be

$$y - y_i = Ax_i + B - y_i \tag{10}$$

To determine the best straight line which fits the N sets of data, A and B have to be chosen so that the sum of the squares of the "errors" is least. This means that the simultaneous equations, obtained by taking the partial derivatives of $(y - y_i)^2$ with respect to A and B, should be solved. This condition leads then to the following results

$$A = \frac{N\sum(x_i y_i) - \sum x_i \sum y_i}{\Delta} \tag{11}$$

and

$$B = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum(x_i y_i)}{\Delta} \tag{12}$$

with

$$\Delta = N\sum x_i^2 - (\sum x_i)^2 \tag{13}$$

In the case that the line goes through the origin (B = 0), it can be shown that

$$A = \frac{\sum(x_i y_i)}{\sum x_i^2} = \frac{\sum y_i}{\sum x_i} \tag{14}$$

The *correlation coefficient r* provides an indicator of how good a fit the best straight line is, This coefficient is defined as

$$r = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum(x_i - \overline{x})^2 \sum(y_i - \overline{y})^2}} \tag{15}$$

For r = 0 the values of x and y are independent of one another and there is no linear correlation. The closer r is to +1 or to –1, the better the linear correlation is.

If the relation between the two variables x and y is not linear, a polynomial of higher order than 1 has to be fitted to the N data points (see reference 1 and 2). The coefficients can then be determined by the least square method as in the case of the linear relationship.

However the following function relations between x and y can be reduced to a linear relation

1. Exponential curve $y = B \exp(Ax)$
2. Logarithmic curve $y = B + A \ln(x)$
3. Power curve $y = B x^A$

The comparison of the above curves with a straight line is summarized in the following table

Table 2

	A	**B**	x	y
Linear $y = Ax + B$	A	B	x	y
Exponential $y = B \exp(Ax)$	A	ln B	x	ln y
Logarithmic $y = B + A \ln(x)$	A	B	ln x	y
Power $y = B x^A$	A	ln B	ln x	ln y

6. REFERENCES

1. Errors of Observation and their Treatment.
   J. Topping, Chapman and Hall (1972), London, UK.
2. An Introduction to Error Analysis.
   J.R. Taylor, Oxford University Press (1982), UK
3. The Statistical Analysis of Experimental Data
   J. Mandel, Dover Publications, Inc.(1984), USA
4. SPSS for Windows
   M.J. Norusis, SPSS Inc. (1993), USA